Sound in Ducts with Flow

A Lorentz transformation there and back – and how to compute wavenumbers, attenuation and transmission loss the easy way

Michael H. Grant
Lektor emeritus in physics & mathematics, Grenaa Gymnasium, Denmark

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Abstract

When the gas in a duct flows with velocity V, the sound pressure no longer satisfies the ordinary wave equation but the convected wave equation, with a mixed term $\partial^2 p/\partial x\,\partial t$. The Galilean transformation that is often used moves the calculation into the rest frame of the gas – but there the ends of the duct and the lined sections move, and a single harmonic oscillation in the laboratory becomes two different frequencies. The wave equation is not invariant under the Galilean transformation. A Lorentz transformation, with the speed of sound c in place of the speed of light, removes the mixed term *and* keeps the walls at rest. One can then solve the familiar problem without flow and transform back. This note goes through the transformation, the hard-walled modes, the transfer matrix of a duct segment, the transmission loss of an expansion chamber, and the lined duct with Myers' boundary condition. All results have been checked numerically against a direct solution in the laboratory frame.

1 The convected wave equation

The gas flows with constant velocity V in the x direction (plug flow), and $M = V/c$ is the Mach number. Linearising the Euler and continuity equations about this mean flow, the time derivatives appear as the *material* derivative $D/Dt = \partial_t + V\partial_x$:

$$\rho_0 \frac{D\mathbf{u}}{Dt} = -\nabla p, \qquad \frac{1}{c^2}\frac{Dp}{Dt} + \rho_0 \nabla\cdot \mathbf{u} = 0. \tag{1}$$

Eliminating $\mathbf{u}$ gives

$$\frac{1}{c^2}\Big(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\Big)^2 p - \nabla^2 p = 0, \tag{2}$$

or, written out in one dimension as on the page *Relativistic acoustics*,

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \frac{2M}{c}\frac{\partial^2 p}{\partial x\,\partial t} - (1-M^2)\frac{\partial^2 p}{\partial x^2} - \nabla_\perp^2 p = 0. \tag{3}$$

Here $\nabla_\perp^2 = \partial_y^2 + \partial_z^2$ is the transverse part. The mixed term makes the solutions asymmetric: sound behaves differently with and against the flow.

Throughout this note the time factor is $e^{i\omega t}$, and a wave $e^{i(\omega t - k_x x)}$ with $\operatorname{Re} k_x > 0$ travels downstream. Also

$$k = \frac{\omega}{c}, \qquad \gamma = \frac{1}{\sqrt{1-M^2}}.$$

2 Why the Galilean transformation is not enough

The obvious step is to move into the rest frame of the gas:

$$x'' = x - Vt, \qquad t'' = t. \tag{4}$$

Then $\partial_t + V\partial_x = \partial_{t''}$ and $\partial_x = \partial_{x''}$, and (2) becomes the d'Alembert equation in (x'', t''). That looks convenient, but has three drawbacks:

1. **The wave equation is not Galilean invariant.** That is exactly why the convected equation (2) appears: applied to the d'Alembert equation in the gas frame, the Galilean transformation produces the mixed term in the laboratory. The *form* of the equation depends on the frame.
2. **The boundaries move.** The duct, the ends of the expansion chamber and the start and end of the lining are fixed in the laboratory, but in the gas frame they lie at $x'' = -Vt''$, $x'' = L - Vt''$ and so on. A problem with fixed boundaries becomes a problem with moving boundaries.
3. **Frequency is not conserved.** A downstream and an upstream wave with the same laboratory frequency ω have the frequencies $\omega'' = \omega - Vk_x$, which differ for the two waves (Doppler). A single harmonic oscillation becomes two.

The Galilean transformation is fine for a single propagation problem without boundaries, and it gives the simple damping model on *Relativistic acoustics*. But for reflections, expansion chambers and lined ducts it is unwieldy.

3 The Lorentz transformation

3.1 There

With the speed of sound c in place of the speed of light, define

$$x' = \gamma x, \qquad t' = \frac{t}{\gamma} + \frac{\gamma M}{c}\, x, \qquad y' = y, \quad z' = z. \tag{5}$$

The chain rule gives

$$\frac{\partial}{\partial t} = \frac{1}{\gamma}\frac{\partial}{\partial t'}, \qquad \frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial x'} + \frac{\gamma M}{c}\frac{\partial}{\partial t'}, \tag{6}$$

and hence

$$\frac{\partial}{\partial t} + V\frac{\partial}{\partial x} = \Big(\frac{1}{\gamma} + \gamma M^2\Big)\frac{\partial}{\partial t'} + \gamma M c\,\frac{\partial}{\partial x'} = \gamma\Big(\frac{\partial}{\partial t'} + Mc\,\frac{\partial}{\partial x'}\Big), \tag{7}$$

because $1/\gamma + \gamma M^2 = \gamma(1 - M^2) + \gamma M^2 = \gamma$. Inserting in (2):

$$\frac{1}{c^2}(\partial_t + V\partial_x)^2 = \gamma^2\Big[\frac{1}{c^2}\partial_{t'}^2 + \frac{2M}{c}\partial_{t'}\partial_{x'} + M^2\partial_{x'}^2\Big], \tag{8}$$

$$\partial_x^2 = \gamma^2\Big[\partial_{x'}^2 + \frac{2M}{c}\partial_{x'}\partial_{t'} + \frac{M^2}{c^2}\partial_{t'}^2\Big]. \tag{9}$$

On subtraction the mixed terms cancel, and $\gamma^2(1 - M^2) = 1$ gives

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t'^2} - \frac{\partial^2 p}{\partial x'^2} - \nabla_\perp^2 p = 0. \tag{10}$$

This is the ordinary wave equation without flow. The crucial point is that the transformation only *stretches* x by the factor γ and does not mix t into the spatial coordinate. The walls $y, z = \text{const.}$ stay at rest, and a duct segment $0 \le x \le L$ becomes $0 \le x' \le \gamma L$.

3.2 Back

The inverse transformation is

$$x = \frac{x'}{\gamma}, \qquad t = \gamma t' - \frac{\gamma M}{c}\, x'. \tag{11}$$

3.3 Harmonic oscillations: frequency is conserved

Write the sound pressure in the laboratory as $p = P(x, y, z)\, e^{i\omega t}$. With the inverse transformation,

$$\omega t = \gamma\omega\, t' - \gamma k M\, x' = \omega' t' - KMx', \qquad \omega' = \gamma\omega, \quad K = \frac{\omega'}{c} = \gamma k. \tag{12}$$

So if we put

$$P(x, y, z) = \Phi(x', y, z)\, e^{\,iKMx'} = \Phi(\gamma x, y, z)\, e^{\,i\gamma^2 kMx}, \tag{13}$$

then $p = \Phi(x', y, z)\, e^{i\omega' t'}$: a pure harmonic oscillation with *one* frequency ω' in the primed frame too. From (10) follows the ordinary Helmholtz equation

$$\frac{\partial^2 \Phi}{\partial x'^2} + \nabla_\perp^2 \Phi + K^2 \Phi = 0. \tag{14}$$

The factor $e^{iKMx'}$ has a real wavenumber. It is a pure *convective phase shift* and never contributes to attenuation. All the attenuation sits in Φ.

Translation table (laboratory → primed frame)

frequency	$\omega \;\to\; \omega' = \gamma\omega, \quad k \;\to\; K = \gamma k$
length along the duct	$L \;\to\; L' = \gamma L$
transverse dimensions, hard walls	unchanged
sound pressure	$P = \Phi\, e^{iKMx'}$
wavenumber back	$k_x^{\pm} = \gamma\,(\pm\kappa - MK)$, where $\Phi \propto e^{\mp i\kappa x'}$

The last line follows directly from (13): $e^{iKMx'} e^{\mp i\kappa x'} = e^{-i\gamma(\pm\kappa - MK)x}$.

4 Hard-walled ducts

4.1 Modes and wavenumbers

In the primed frame a mode is $\Phi = \psi_{mn}(y, z)\, e^{\mp i\kappa_{mn} x'}$ with $\nabla_\perp^2 \psi_{mn} = -k_{mn}^2 \psi_{mn}$ and $\kappa_{mn} = \sqrt{K^2 - k_{mn}^2}$, exactly as without flow. Back in the laboratory:

$$k_x^{\pm} = \gamma(\,\pm\kappa_{mn} - MK) = \frac{-Mk \pm \sqrt{k^2 - (1 - M^2)k_{mn}^2}}{1 - M^2}. \tag{15}$$

Plane wave $(k_{mn} = 0,\ \kappa = K)$:

$$k_x^{+} = \gamma^2 k(1 - M) = \frac{k}{1 + M}, \qquad k_x^{-} = -\gamma^2 k(1 + M) = -\frac{k}{1 - M}. \tag{16}$$

The phase speeds are $c + V$ and $c - V$, as they should be.

Cut-on frequency. Mode (m, n) propagates when $K > k_{mn}$, i.e.

$$f > f_{mn}\sqrt{1 - M^2}, \tag{17}$$

where $f_{mn} = ck_{mn}/2\pi$ is the cut-on frequency without flow. The flow lowers all cut-on frequencies by the factor $1/\gamma$, equally for both directions. For a rectangular duct of width b, $f_{10} = \frac{c}{2b}\sqrt{1 - M^2}$.

4.2 Particle velocity and intensity

The momentum equation for a plane wave $p_\pm e^{i(\omega t - k_x^\pm x)}$ gives $\rho_0\, i(\omega - Vk_x)u = ik_x p$. Downstream, $\omega - Vk_x^+ = \omega/(1 + M)$ and $k_x^+ = k/(1 + M)$, so

$$u_\pm = \pm\frac{p_\pm}{\rho_0 c}. \tag{18}$$

The characteristic impedance is unchanged. The transported power, however, is (Morfey 1971)

$$W_\pm = \frac{S\,|p_\pm|^2}{2\rho_0 c}\,(1 \pm M)^2, \tag{19}$$

with amplitude values. When inlet and outlet ducts have the same area and the same Mach number, the factor $(1 \pm M)^2$ cancels in the transmission loss.

4.3 Transfer matrix of a duct segment – there and back

In the primed frame a duct segment of length $L' = \gamma L$ is the familiar segment without flow. Back in the laboratory, using (13) and $u_\pm = \pm p_\pm/\rho_0 c$,

$$p(x) = e^{i\gamma^2 kMx}\big(A\,e^{-i\gamma^2 kx} + B\,e^{i\gamma^2 kx}\big), \tag{20}$$

$$\rho_0 c\,u(x) = e^{i\gamma^2 kMx}\big(A\,e^{-i\gamma^2 kx} - B\,e^{i\gamma^2 kx}\big). \tag{21}$$

With the volume velocity $q = Su$ and $Y = \rho_0 c/S$ the transfer matrix becomes

$$\begin{pmatrix} p \\ q \end{pmatrix}_{x=0} = e^{-i\gamma^2 kML} \begin{pmatrix} \cos\gamma^2 kL & iY\sin\gamma^2 kL \\ iY^{-1}\sin\gamma^2 kL & \cos\gamma^2 kL \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}_{x=L}. \tag{22}$$

This is the no-flow matrix with two changes:

- the electrical length kL is replaced by $\gamma^2 kL = kL/(1 - M^2)$;
- there is a common phase factor $e^{-i\gamma^2 kML}$ of modulus 1.

That is the whole gain of going there and back: nothing new has to be solved.

5 The expansion chamber

Inlet and outlet ducts have area S_1 and Mach number M_1. The chamber has area $S_2 = mS_1$ and length L. Continuity of the mean flow gives the Mach number in the chamber, $M_k = M_1/m$. At the two area changes, continuity of acoustic pressure and acoustic volume velocity is assumed. The transmission loss for an anechoic termination is

$$\mathrm{TL} = 20\log_{10}\Big|\tfrac{1}{2}(T_{11} + T_{12}/Y_1 + Y_1 T_{21} + T_{22})\Big|. \tag{23}$$

The phase factor in (22) has modulus 1 and drops out. The power factor $(1 + M_1)^2$ is the same in inlet and outlet. What remains is the classical formula, merely with $kL \to kL/(1 - M_k^2)$:

$$\mathrm{TL} = 10\log_{10}\Big[1 + \tfrac{1}{4}\Big(m - \frac{1}{m}\Big)^2 \sin^2\Big(\frac{kL}{1-M_k^2}\Big)\Big]. \tag{24}$$

The maximum attenuation, $10\log_{10}[1+\frac{1}{4}(m-1/m)^2]$, is unchanged, but the peaks move down in frequency to

$$f_n = \frac{(2n-1)\,c\,(1-M_k^2)}{4L}, \qquad n = 1, 2, \ldots \tag{25}$$

and the zeros correspondingly to $f = n\,c(1-M_k^2)/2L$. Because $M_k = M_1/m$ is small in a large chamber, the effect is modest (figure 1).

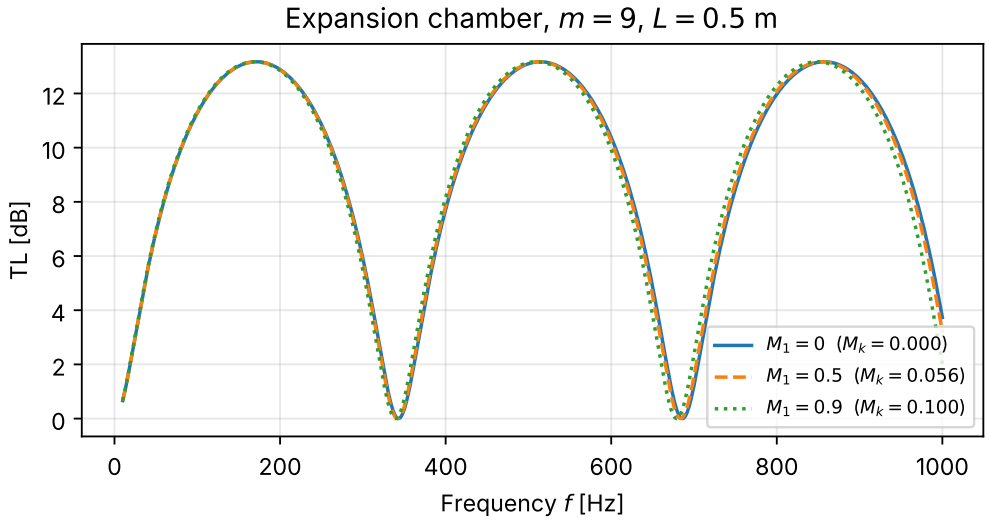


Figure 1: Transmission loss of an expansion chamber with $m = 9$ and $L = 0.5\,\mathrm{m}$ from (24). The curves were computed both with the transfer matrix (22) and with the closed formula. They agree to $10^{-9}\,\mathrm{dB}$.

Caveat. With flow, the simple junction conditions are only an approximation. At a sudden area change the flow separates. Munjal (2014) therefore uses continuity of mass flux and a pressure loss with a loss coefficient. This adds terms of order M_1, which matter mainly for $M_1 \gtrsim 0.2$. The transformation does not change this: the junction conditions are physics at the discontinuity, not part of the wave equation.

6 The lined duct

6.1 Myers' boundary condition

For a locally reacting lining with normalised impedance $\zeta = Z/\rho_0 c$ (for $e^{i\omega t}$) and plug flow right up to the wall, continuity of particle *displacement* is required (Ingard 1959, Myers 1980). The gas flows past a moving wall, so the normal velocity in the gas is $(\partial_t + V\partial_x)\xi$, while that of the wall is $\partial_t \xi = p/Z$. For a mode $e^{i(\omega t - k_x x)}$ this gives

$$v_n = \Big(1 - \frac{M k_x}{k}\Big)\frac{p}{Z}. \tag{26}$$

For a duct of height $2h$, lined on both walls, and a symmetric mode $p \propto \cos(qy)$, the momentum equation in the y direction gives

$$q\tan(qh) = \frac{ik}{\zeta}\Big(1 - \frac{Mk_x}{k}\Big)^2, \qquad (k - Mk_x)^2 = k_x^2 + q^2. \tag{27}$$

Solved directly, this is a coupled, non-linear problem in (k_x, q).

6.2 There: the lined duct without flow

In the primed frame q is unchanged (the transverse coordinate is not transformed), and $\kappa = \sqrt{K^2 - q^2}$. With $k_x^{\pm} = \gamma(\pm\kappa - MK)$,

$$k - Mk_x^{\pm} = \frac{K}{\gamma} - \gamma M(\pm\kappa - MK) = \gamma\,(K \mp M\kappa). \tag{28}$$

The boundary condition (27) therefore becomes

$$q\tan(qh) = \frac{iK}{\zeta'_{\pm}}, \qquad \zeta'_{\pm} = \frac{\zeta}{\gamma^3(1 \mp M\kappa/K)^2}. \tag{29}$$

This is *exactly* the boundary condition of a lined duct without flow, at the frequency $\omega' = \gamma\omega$ and with an effective impedance $\zeta'_{\pm}$. One can reuse one's routine or one's curves from the no-flow case. The only complication is that ζ' depends on the mode's own κ, so one iterates: start with $\kappa = K$, solve, update ζ', solve again. It converges in a few steps.

6.3 Back: the attenuation

Because the phase factor $e^{iKMx'}$ is purely imaginary in the exponent, the attenuation in the laboratory is

$$\alpha_{\pm} = -\operatorname{Im} k_x^{\pm} = \gamma\,|\operatorname{Im}\kappa|, \qquad D_{\pm} = 8.686\,\alpha_{\pm}\ \text{ dB/m}, \tag{30}$$

and a lined section of length L gives, neglecting reflections at the ends of the lining,

$$\text{TL} \approx 8.686\,\alpha_{\pm} L\ \text{dB}. \tag{31}$$

6.4 A closed result: weak lining

If the admittance of the lining is small, $|1/\zeta| \ll kh$, then $qh \ll 1$ and $\kappa \approx K$. Then $q^2 \approx iK/(\zeta' h)$ and $\kappa \approx K - q^2/2K$, i.e. $\operatorname{Im}\kappa \approx -\operatorname{Re}(1/\zeta')/2h$. With (29) and $\kappa \approx K$:

$$\alpha_{\pm} = \gamma \cdot \gamma^3(1 \mp M)^2\,\frac{\operatorname{Re}(1/\zeta)}{2h} = \frac{(1 \mp M)^2}{(1 - M^2)^2}\,\alpha_0 \tag{32}$$

$$\alpha_{\pm} = \frac{\alpha_0}{(1 \pm M)^2}, \qquad \frac{\alpha_-}{\alpha_+} = \Big(\frac{1+M}{1-M}\Big)^2. \tag{33}$$

Here $\alpha_0 = \operatorname{Re}(1/\zeta)/2h$ is the attenuation without flow. Compare with the result $\alpha_0/(1 \pm M)$ on the page *Relativistic acoustics*. There the damping is regarded as a property of the gas, acting at a fixed rate per unit time. With a lining, the boundary condition gives an extra factor $1/(1 \pm M)$: the displacement of the wall is seen Doppler-shifted by the flowing gas, and the factor $(1 - Mk_x/k) = 1/(1 \pm M)$ enters squared in (27). At $M = 0.3$ the ratio of upstream to downstream attenuation is therefore 3.45 with a lining, against 1.86 in the gas model.

6.5 Numerical example: porous lining

Figure 2 shows the least attenuated mode in a duct with $2h = 0.2\,\mathrm{m}$, lined on both sides with $d = 5\,\mathrm{cm}$ of porous material with flow resistivity $\sigma = 10\,\mathrm{kPa\,s/m^2}$ on a rigid backing. The impedance is computed with Delany–Bazley. Each curve is solved in the primed frame and transformed back. The solution was checked by inserting it in (27) in the laboratory frame; the residual is below 10^{-6} everywhere.

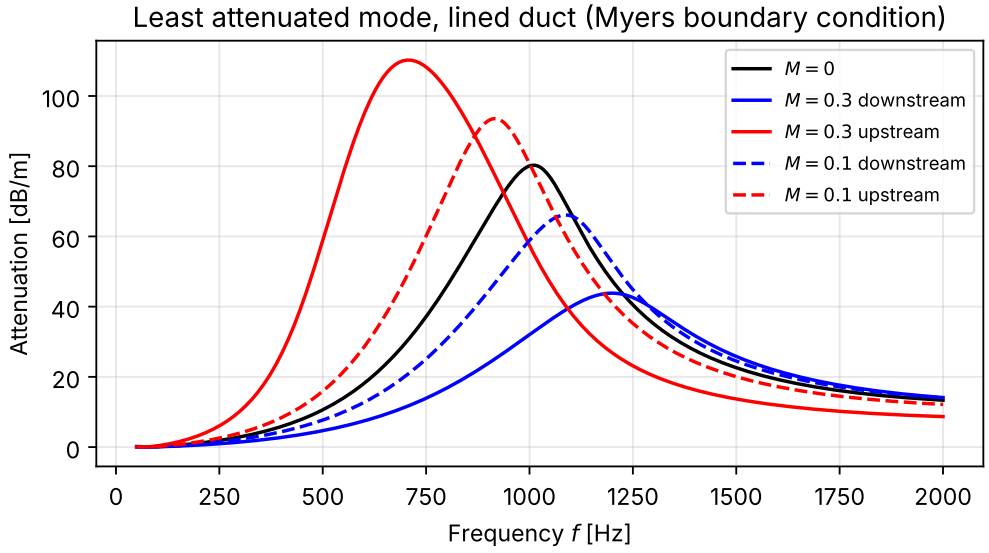


Figure 2: Attenuation of the least attenuated mode with and against the flow. The peak moves down in frequency upstream and up downstream.

Two things are seen:

1. At low frequencies the duct attenuates most upstream, as expected.
2. The attenuation peak *moves*. This follows directly from the transformation: in the primed frame the frequency is $\gamma\omega$, and the effective impedance $\zeta'_\pm$ is scaled by $1/\gamma^3(1 \mp M)^2$. Upstream the lining becomes effectively "softer" and the peak lies lower. Above the peak the attenuation can therefore be *smaller* upstream than without flow.

f [Hz]	$M = -0.3$	$M = -0.1$	$M = 0$	$M = +0.1$	$M = +0.3$
125	0.8	0.4	0.3	0.2	0.2
250	6.0	2.6	1.9	1.5	0.9
500	58.8	15.9	10.6	7.7	4.7
1000	57.3	80.7	80.1	58.9	32.0
2000	8.7	12.2	13.4	14.1	14.1

Table 1: Attenuation in dB per metre of lined duct (least attenuated mode). Negative M means upstream. At 2000 Hz the least attenuated mode has drawn in towards the centre of the duct and is only weakly attenuated; here the attenuation is largest *downstream*, because the peak has moved. Values above about 40 dB are theoretical: in practice the transmission loss is limited by higher modes and flanking transmission.

Figure 3 compares the ratio $\alpha(M)/\alpha(0)$ with the two closed expressions. For a weak lining ($\zeta = 100$) the numerical points lie exactly on $1/(1 \pm M)^2$, confirming (33). For the realistic

porous lining the flow effect is even larger, because κ is not close to K, and because the lining's own resonance moves.

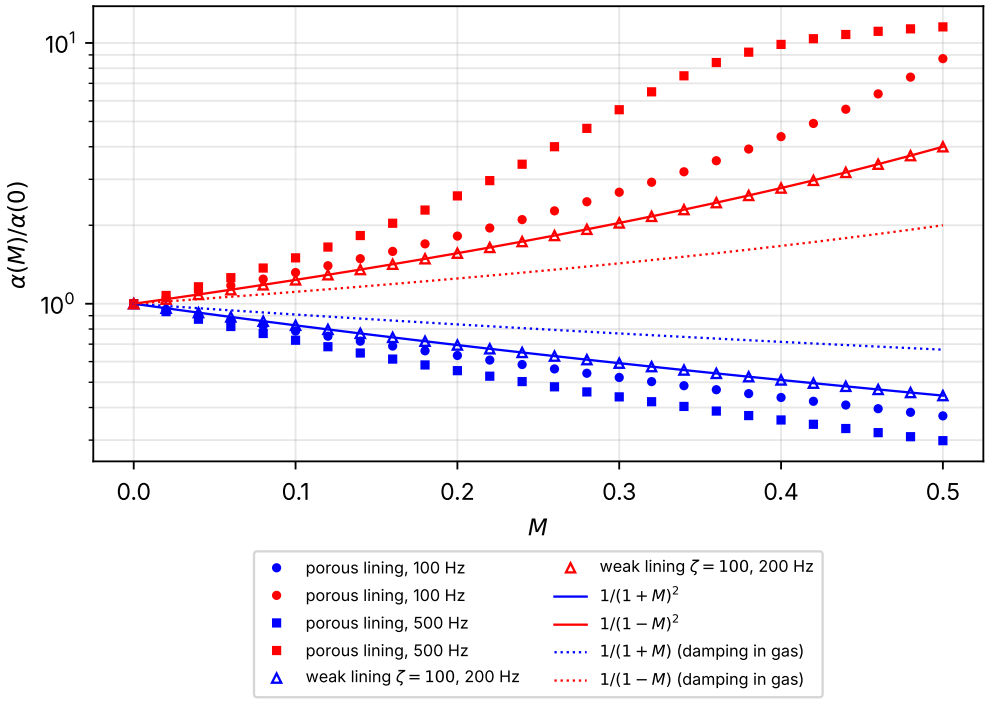


Figure 3: $\alpha(M)/\alpha(0)$ downstream (blue) and upstream (red). Triangles: weak, purely resistive lining, following (33). Circles and squares: the porous lining at 100 and 500 Hz. Dotted lines: the gas model $1/(1 \pm M)$.

Caveat. Myers' boundary condition assumes plug flow right up to the wall. A real boundary layer of finite thickness weakens the influence of the flow on the boundary condition, especially at high frequencies. The Myers condition therefore gives an upper bound on the asymmetry.

7 The recipe – there and back

1. **The problem in the laboratory:** frequency ω, lengths L, cross-section, Mach number M, lining impedance ζ.
2. **There:** $\omega' = \gamma\omega$ $(K = \gamma k)$, $L' = \gamma L$, transverse dimensions unchanged, lining $\zeta \to \zeta'_\pm = \zeta/\gamma^3(1 \mp M\kappa/K)^2$.
3. **Solve** the usual problem without flow: modes κ, transfer matrices, reflections.
4. **Back:** multiply by $e^{iKMx'}$. This gives $k_x^\pm = \gamma(\pm\kappa - MK)$, $kL \to kL/(1-M^2)$ in transfer matrices, and the attenuation $\alpha = \gamma|\operatorname{Im}\kappa|$.
5. **Power:** use $W_\pm \propto (1 \pm M)^2|p_\pm|^2$ if inlet and outlet ducts have different Mach numbers.

8 Summary

Quantity	Without flow	With flow (via Lorentz)
Plane wave, wavenumber	$\pm k$	$k/(1+M),\ \ -k/(1-M)$
Cut-on frequency	f_{mn}	$f_{mn}\sqrt{1-M^2}$
Duct segment, electrical length	kL	$kL/(1-M^2)$ (+ phase factor)
Expansion chamber	$\sin^2 kL$	$\sin^2(kL/(1-M_k^2))$
Attenuation, gas model	α_0	$\alpha_0/(1\pm M)$
Attenuation, weak lining (Myers)	α_0	$\alpha_0/(1\pm M)^2$
Lining, general	ζ	$\zeta/\gamma^3(1\mp M\kappa/K)^2$ at $\gamma\omega$

The Lorentz transformation preserves the wave equation and keeps the geometry at rest. Therefore almost all duct acoustics with uniform flow can be read directly from the known theory without flow. The price is a stretch of lengths by γ, a frequency scaled by γ, a convective phase factor of no consequence for the attenuation and, for lined ducts, an effective impedance.

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- Interactive 3D simulations for this note: mhgfysik.dk/lorentz-roer-en.html.